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On Monotonicities of the Generalized Mean Ratio and Related Results

LEON JAY GLESER*

Johns Hopkins University, Baltimore, Maryland

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1. INTRODUCTION AND SUMMARY

The purpose of this paper is to discuss lower (and upper) bounds for ratios of weighted means M_r, $-\infty \leqslant r \leqslant \infty$, of positive variables $x_1, x_2, \ldots, x_n$ in terms of corresponding ratios of weighted means for any subset $\{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$, $m < n$, of these values. Some of the results obtained are extended to a more abstract context.

2. A MONOTONICITY OF THE GENERALIZED MEAN RATIO

Let $p_1, p_2, \ldots, p_n$ $(n > 1)$ be positive numbers with $\sum_{i=1}^n p_i = 1$. For every collection $x_1, x_2, \ldots, x_n$ of positive real numbers, and for every real v, define the v-th weighted mean M_v to be

$$M_v = \left(\sum_{i=1}^n p_i x_i^v\right)^{1/v}, \tag{2.1}$$

where it is understood that

$$\lim_{v\to\infty} M_v = \max_{1\leqslant i\leqslant n} x_i, \qquad \lim_{v\to-\infty} M_v = \min_{1\leqslant i\leqslant n} x_i,$$

and

$$\lim_{v\to 0} M_v = \prod_{i=1}^n x_i^{p_i}.$$

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The *generalized mean ratio* $T_{r,s}$ is defined to be

$$T_{r,s}(n) = \frac{M_r}{M_s}, \qquad r > s. \tag{2.2}$$

The well known *moment inequality* (viz., [1]) tells us that $T_{r,s}(n) \geqslant 1$, with strict inequality unless $x_1 = x_2 = \cdots = x_n$.

Suppose that we eliminate one of the x_i's (say, x_{j_0}) and renormalize the probabilities p_i, $i \neq j_0$, so that they sum to one. Form the reduced generalized mean ratio

$$T_{r,s}(n-1; j_0) = \left[\frac{\sum_{i\neq j_0} p_i x_i^r}{\sum_{i\neq j_0} p_i}\right]^{1/r} \left[\frac{\sum_{i\neq j_0} p_i x_i^s}{\sum_{i\neq j_0} p_i}\right]^{-1/s} \tag{2.3}$$

from the remaining x_i's, $i \neq j_0$. It might be hypothesized that $T_{r,s}(n) \geqslant T_{r,s}(n-1; j_0)$ for every j_0 (i.e., that the generalized mean ratio is monotone in n). This hypothesis is false in general; however, we can verify the weaker assertion: $T_{r,s}(n) \geqslant T_{r,s}(n-1; j)$, for some j.

THEOREM 2.1. *If* $r > s$,

$$T_{r,s}(n) \geqslant \min_{1\leqslant j\leqslant n} T_{r,s}(n-1; j). \tag{2.4}$$

PROOF. Let

$$q_j = \sum_{i\neq j} p_i, \qquad w_j^s = q_j^{-1} \sum_{i\neq j} p_i x_i.$$

Then

$$T_{r,s}(n) = \frac{\left[(n-1)^{-1} \sum_{j=1}^{n} q_j w_j^r (T_{r,s}(n-1; j))^r\right]^{1/r}}{\left[(n-1)^{-1} \sum_{j=1}^{n} q_j w_j^s\right]^{1/s}}$$

$$\geqslant [\min_{1<j\leqslant n} T_{r,s}(n-1; j)] \frac{\left\{(n-1)^{-1} \sum_{j=1}^{n} q_j w_j^r\right\}^{1/r}}{\left\{(n-1)^{-1} \sum_{j=1}^{n} q_j w_j^s\right\}^{1/s}}. \tag{2.5}$$

But $(n-1)^{-1}\sum_{j=1}^{n} q_j = 1$, so that the result (2.4) follows from the moment inequality

$$\left\{(n-1)^{-1}\sum_{j=1}^{n} q_j w_j^{\,r}\right\}^{1/r} \geqslant \left\{(n-1)^{-1}\sum_{j=1}^{n} q_j w_j^{\,s}\right\}^{1/s}. \quad \|$$

We might also ask whether there is always a j_0 such that $T_{r,s}(n-1; j_0) \geqslant T_{r,s}(n)$. In general, this assertion is false. The following easily verified condition, however, is sufficient for its validity.

THEOREM 2.2. *Let $x_{[1]} \leqslant x_{[2]} \leqslant \cdots \leqslant x_{[n]}$ be the ordered values of the x_i's. If $r > s$, and if for some k, $1 \leqslant k \leqslant n$, we have*

$$M_s \leqslant x_{[k]} \leqslant M_r, \tag{2.6}$$

then

$$T_{r,s}(n) \leqslant \max_{1 \leqslant j \leqslant n} T_{r,s}(n-1; j).$$

Proof. Let $p_{[i]}$ be the probability associated with $x_{[i]}$, $i = 1, 2,..., n$. From (2.6),

$$M_r^{\,r} \leqslant \Big(\sum_{i \neq k} p_{[i]} x_{[i]}^{r}\Big) + p_{[k]} M_r^{\,r},$$

$$M_s^{\,s} \geqslant \Big(\sum_{i \neq k} p_{[i]} x_{[i]}^{s}\Big) + p_{[k]} M_s^{\,s}.$$

Thus

$$M_r \leqslant \Big((1-p_{[k]})^{-1} \sum_{i \neq k} p_{[i]} x_{[i]}^{r}\Big)^{1/r} \equiv M_r([k]),$$

$$M_s \geqslant \Big((1-p_{[k]})^{-1} \sum_{i \neq k} p_{[i]} x_{[i]}^{s}\Big)^{1/s} \equiv M_s([k]),$$

and consequently

$$T_{r,s} = \frac{M_r}{M_s} \leqslant \frac{M_r([k])}{M_s([k])} \leqslant \max_{1 \leqslant j \leqslant n} T_{r,s}(n-1; j). \quad \|$$

Similar results can be obtained for comparing the generalized mean ratio $T_{r,s}(n)$ with reduced generalized mean ratios formed from arbitrary subcollections $\{x_{j_1}, x_{j_2},..., x_{j_m}\}$, $m < n$, of $\{x_1, x_2,..., x_n\}$. Such assertions, and many other statements of interest to us, can be very conveniently obtained as special cases of results found in the generalization of our problem to a more abstract, measure-theoretic context.

3. Monotonicities of the Generalized Mean Ratio on Arbitrary Probability Spaces

Consider the probability space $(\mathscr{X}, \mathscr{B}, P)$, where P is a probability measure defined on the σ-algebra $\mathscr{B}$ of subsets of $\mathscr{X}$. Consider any positive measurable function f, and define the v-th moment of f to be

$$\mu_v(\mathscr{X}) = \int_{\mathscr{X}} f^v(x)\, dP(x), \qquad -\infty \leqslant v \leqslant \infty. \tag{3.1}$$

Then the generalized mean ratio defined with repect to f is

$$T_{r,s}(\mathscr{X}) \equiv \frac{\mu_r^{1/r}}{\mu_s^{1/s}}, \qquad r > s. \tag{3.2}$$

We note that $\lim_{r\to\infty} \mu_r^{1/r}$ can be taken to be the essential supremum of $f(x)$ over $\mathscr{X}$ and that $\lim_{s\to-\infty} \mu_s^{1/s}$ can be taken to be the essential infimum of f over $\mathscr{X}$. Further, $\lim_{r\to 0} \mu_r^{1/r}$ equals $\exp\{\int_{\mathscr{X}} \log f(x)\, dP(x)\}$. Note also that the moment inequality in this context states that $T_{r,s}(\mathscr{X}) \geqslant 1$, with equality holding if and only if f is essentially (almost surely) constant.

Consider, now, any subset $A \in \mathscr{B}$ of $\mathscr{X}$. Renormalize the probability measure P over A so that the resulting measure is again a probability measure. Such normalization yields the conditional probability measure:

$$P(B \mid A) \equiv \begin{cases} P(B \cap A)/P(A), & \text{if} \quad P(A) > 0, \\ 0, & \text{if} \quad P(A) = 0, \end{cases} \tag{3.3}$$

all $B \in \mathscr{B}$. Define the vth conditional moment of f to be

$$\mu_v(A) = \int_A f^v(x)\, dP(x \mid A), \tag{3.4}$$

and define the generalized mean ratio of f with respect to $P(\cdot \mid A)$ to be

$$T_{r,s}(A) = \frac{\mu_r^{1/r}(A)}{\mu_s^{1/r}(A)}, \qquad r > s. \tag{3.5}$$

The discrete case discussed in Section 2 is a special case of the above model. For, if $\mathscr{X} = \{x_1, x_2, ..., x_n\}$, P is the measure putting mass p_i on i,

$i = 1, 2, \ldots, n$, $\mathscr{B}$ is the collection of all subsets $A = \{j_1, j_2, \ldots, j_m\}$ of $\mathscr{X}$, and $f(i) = x_i$, $i = 1, 2, \ldots, n$, then $T_{r,s}(\mathscr{X})$ corresponds to $T_{r,s}(n)$. Further, if $A_j = \{i : i \neq j\}$, then $T_{r,s}(A_j) = T_{r,s}(n-1; j)$.

We have already seen in Section 2 that $T_{r,s}(\mathscr{X})$ need not exceed $T_{r,s}(A)$ for $A \in \mathscr{B}$. The assertion that $T_{r,s}(\mathscr{X}) \geqslant \min_{A \in \mathscr{B}} T_{r,s}(A)$ is meaningless if we allow $\mathscr{B}$ to contain point sets $A_\epsilon = \{x : f(x) = \epsilon\}$. For example, if for some ϵ_0, $P(A_{\epsilon_0}) > 0$, the assertion $T_{r,s}(\mathscr{X}) \geqslant \min_{A \in \mathscr{B}} T_{r,s}(A)$ becomes $T_{r,s}(\mathscr{X}) \geqslant T_{r,s}(A_{\epsilon_0}) = 1$, which tells us no more than the moment inequality. We can, however, ask for sufficient conditions for $T_{r,s}(\mathscr{X}) \geqslant T_{r,s}(A)$, given $A \in \mathscr{B}$.

Lemma 3.1. *If $r > s$, $A \in \mathscr{B}$, then*

$$\min\{T_{r,s}(A), T_{r,s}(A^c)\} \leqslant T_{r,s}(\mathscr{X}) \leqslant [\max\{T_{r,s}(A), T_{r,s}(A^c)\}] \frac{[p + (1-p)\gamma^r]^{1/r}}{[p + (1-p)\gamma^s]^{1/s}}, \tag{3.6}$$

where A^c is the complement of A in $\mathscr{X}$, $p = P(A) = 1 - P(A^c)$, and $\gamma^s = \mu_s(A^c)/\mu_s(A)$.

Proof. Note that

$$T_{r,s}(\mathscr{X}) = \frac{[p\mu_r(A) + (1-p)\mu_r(A^c)]^{1/r}}{[p\mu_s(A) + (1-p)\mu_s(A^c)]^{1/s}} = \frac{[p(T_{r,s}(A))^r + (1-p)\gamma^r(T_{r,s}(A^c))^r]^{1/r}}{[p + (1-p)\gamma^s]^{1/s}}. \tag{3.7}$$

Since the moment inequality tells us that

$$t(\gamma) \equiv \frac{(p + (1-p)\gamma^r)^{1/r}}{(p + (1-p)\gamma^s)^{1/s}} \geqslant 1,$$

and since

$$t(\gamma)\min\{T_{r,s}(A), T_{r,s}(A^c)\} \leqslant T_{r,s}(\mathscr{X}) \leqslant t(\gamma)\max\{T_{r,s}(A), T_{r,s}(A^c)\},$$

the result (3.6) follows. ||

Theorem 3.2. *If $r > s$, $A \in \mathscr{B}$, then*

$$T_{r,s}(A^c) \geqslant T_{r,s}(A) \Rightarrow T_{r,s}(\mathscr{X}) \geqslant T_{r,s}(A), \tag{3.8}$$

$$T_{r,s}(A^c) \geqslant \Delta_\gamma^* T_{r,s}(A) \Leftrightarrow T_{r,s}(\mathscr{X}) \geqslant T_{r,s}(A), \tag{3.9}$$

where Δ^* is the solution of

$$(p + (1-p)\gamma^r(\Delta_\gamma^*)^r)^{1/r} = (p + (1-p)\gamma^s)^{1/s}. \tag{3.10}$$

PROOF. The result (3.8) is a direct consequence of Lemma 3.1. To prove (3.9), note that from (3.7) we have

$$T_{r,s}(\mathscr{X}) = T_{r,s}(A)\,\frac{[p+(1-p)\,\gamma^r\Delta^r]^{1/r}}{[p+(1-p)\,\gamma^s]^{1/s}}\,,$$

where $\Delta = T_{r,s}(A^c)/T_{r,s}(A)$. Thus $T_{r,s}(\mathscr{X})$ exceeds $T_{r,s}(A)$ if and only if

$$\frac{[p+(1-p)\,\gamma^r\Delta^r]^{1/r}}{[p+(1-p)\,\gamma^s]^{1/s}} \geqslant 1. \tag{3.11}$$

It is easy to see that (3.11) holds if and only if $\Delta \geqslant \Delta_\gamma^*$, where Δ_γ^* is the solution of (3.10). ||

4. A General Bound

In Section 3, our investigations of the relations between $T_{r,s}(\mathscr{X})$ and $T_{r,s}(A)$ require us to know $T_{r,s}(A^c)$ and $\gamma^s = \mu_s(A^c)/\mu_s(A)$. Since $T_{r,s}(A^c)$ and γ^s depend not only on A and $P(\cdot)$ for their values, but also on the nature of f, the results of Section 3 may be difficult to apply.

If instead of asking for a bound on $T_{r,s}(\mathscr{X})$ of the form $T_{r,s}(\mathscr{X}) \geqslant T_{r,s}(A)$, we ask for a bound of the form

$$T_{r,s}(\mathscr{X}) \geqslant h_p(T_{r,s}(A)) \geqslant 1,$$

where $h_p(u)$ is a function defined on $[1, \infty)$ with form depending on A only through $p \equiv P(A)$, the following lemma provides one solution to our problem.

LEMMA 4.1. *For $r > s$,*

$$T_{r,s}(\mathscr{X}) \geqslant \frac{T_{r,s}(A)}{[p+(1-p)\,(T_{r,s}(A))^t]^{1/t}} \geqslant 1, \tag{4.1}$$

where $t = rs/(r-s)$.

PROOF. From Eq. (3.7) and the fact that $T_{r,s}(A^c) \geqslant 1$, we have

$$T_{r,s}(\mathscr{X}) \geqslant \frac{[p\mu_r(A)+(1-p)\,x^r]^{1/r}}{[p\mu_s(A)+(1-p)\,x^s]^{1/s}}\,, \tag{4.2}$$

where $x^s = \mu_s(A^c)$. It is easy to verify (4.1) for $p = P(A) = 0, 1$. Consequently, we can assume $0 < p < 1$ in what follows. Differentiating the right-hand expression with respect to x, we find that the sign of the derivative is equal to the sign of $x^{r-s} - (\mu_r(A))^r\,(\mu_s(A))^{-s}$. Thus $T_{r,s}(\mathscr{X})$ is minimized only at $(x^*)^{r-s} = (\mu_r(A))^r\,(\mu_s(A))^{-s}$. Substituting x^* in (4.2) gives us (4.1). ||

COROLLARY 4.2. *For $r > s$, any j, $1 \leqslant j \leqslant n$,*

$$T_{r,s}(n) \geqslant \frac{T_{r,s}(n-1;j)}{(q_j + p_j(T_{r,s}(n-1;j))^t)^{1/t}}, \tag{4.3}$$

where

$$t = \frac{rs}{r-s}, \qquad q_j = \sum_{i \neq j} p_i .$$

If $r > 0$, $s \leqslant 0$, we can obtain a somewhat more appealing lower bound for $T_{r,s}(\mathscr{X})$ in terms of $T_{r,s}(A)$ and $p = P(A)$.

LEMMA 4.3. *For $r > 0$, $s \leqslant 0$, $A \in \mathscr{B}$,*

$$T_{r,s}(\mathscr{X}) \geqslant T_{r,s}^{p}(A) \geqslant 1. \tag{4.4}$$

PROOF. If $r > 0$, $s \leqslant 0$, then $t = rs/r - s \leqslant 0$. Consequently, the moment inequality tells us that

$$[p + (1-p)\,(T_{r,s}(A))^t]^{1/t} \leqslant [T_{r,s}(A)]^{(1-p)}. \tag{4.5}$$

Equation (4.4) now follows from (4.3) and (4.4). ||

COROLLARY 4.4. *For $r > 0$, $s \leqslant 0$, any j, $1 \leqslant j \leqslant n$,*

$$T_{r,s}(n) \geqslant (T_{r,s}(n-1;j))^{q_j}, \tag{4.6}$$

where

$$q_j = \sum_{i \neq j} p_i .$$

REMARK. If $r > s$, $rs > 0$, the results (4.5) and (4.6) do not, in general, hold.

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REFERENCES

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